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TAUBERIAN THEOREMS FOR THE WEIGHTED MEAN METHOD OF SUMMABILITY OF INTEGRALS

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Abstract. Let q be a positive weight function on $\mathbf{R}_+ := [0, \infty)$ which is integrable in Lebesgue's sense over every finite interval $(0, x)$ for $0 < x < \infty$, in symbol: $q \in L^1_{loc}(\mathbf{R}_+)$ such that $Q(x) := \int_0^x q(t)dt \neq 0$ for each $x > 0$, $Q(0) = 0$ and $Q(x) \rightarrow \infty$ as $x \rightarrow \infty$. Given a real or complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$, we define $s(x) := \int_0^x f(t)dt$ and

$$\tau_q^{(0)}(x) := s(x), \tau_q^{(m)}(x) := \frac{1}{Q(x)} \int_0^x \tau_q^{(m-1)}(t)q(t)dt \quad (x > 0, m = 1, 2, \dots),$$

provided that $Q(x) > 0$.

We say that $\int_0^\infty f(x)dx$ is summable to L by the m -th iteration of weighted mean method determined by the function $q(x)$, or for short, (\overline{N}, q, m) integrable to a finite number L if

$$\lim_{x \rightarrow \infty} \tau_q^{(m)}(x) = L.$$

In this case, we write $s(x) \rightarrow L(\overline{N}, q, m)$.

It is known that if the limit $\lim_{x \rightarrow \infty} s(x) = L$ exists, then $\lim_{x \rightarrow \infty} \tau_q^{(m)}(x) = L$ also exists. However, the converse of this implication is not always true. Some suitable conditions together with the existence of the limit $\lim_{x \rightarrow \infty} \tau_q^{(m)}(x)$, which is so called Tauberian conditions, may imply convergence of $\lim_{x \rightarrow \infty} s(x)$.

In this paper, one- and two-sided Tauberian conditions in terms of the generating function and its generalizations for (\overline{N}, q, m) summable integrals of real- or complex-valued functions have been obtained. Some classical type Tauberian theorems given for Cesàro summability $(C, 1)$ and weighted mean method of summability (\overline{N}, q) have been extended and generalized.

Keywords: Tauberian conditions; weight function; summable integrals; finite interval.

1. Introduction

Let q be a positive weight function on $\mathbf{R}_+ := [0, \infty)$ which is integrable in Lebesgue's sense over every finite interval $(0, x)$ for $0 < x < \infty$, in symbol: $q \in$

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$L^1_{loc}(\mathbf{R}_+)$ such that $Q(x) := \int_0^x q(t)dt \neq 0$ for each $x > 0$, $Q(0) = 0$ and $Q(x) \rightarrow \infty$ as $x \rightarrow \infty$. Given a real or complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$, we define $s(x) := \int_0^x f(t)dt$ and

$$\tau_q^{(0)}(x) := s(x), \tau_q^{(m)}(x) := \frac{1}{Q(x)} \int_0^x \tau_q^{(m-1)}(t)q(t)dt \quad (x > 0, m = 1, 2, \dots),$$

provided that $Q(x) > 0$.

For each integer $m \geq 0$, we define $v_q^{(m)}(x)$ by

$$v_q^{(m)}(x) = \begin{cases} \frac{Q(x)}{q(x)} f(x) & , m = 0 \\ \frac{1}{Q(x)} \int_0^x f(t)Q(t)dt & , m = 1 \\ \frac{1}{Q(x)} \int_0^x v_q^{(m-1)}(t)q(t)dt & , m \geq 2. \end{cases}$$

The identity

$$(1.1) \quad \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) = v_q^{(m)}(x)$$

is known as the weighted Kronecker identity for the weighted mean method of summability.

It is clear from (1.1) that

$$\frac{Q(x)}{q(x)} \frac{d}{dx} \tau_q^{(m)}(x) = v_q^{(m)}(x)$$

for each integer $m \geq 0$ (see [14]). Here, we call $v_q^{(m)}(x)$ the generator of $\tau_q^{(m-1)}(x)$ for each integer $m \geq 1$.

We say that $\int_0^\infty f(x)dx$ is summable to L by the m -th iteration of weighted mean method determined by the function $q(x)$, or for short, (\overline{N}, q, m) summable to a finite number L if

$$(1.2) \quad \lim_{x \rightarrow \infty} \tau_q^{(m)}(x) = L.$$

It is obvious that (\overline{N}, q, m) summability reduces to the ordinary convergence for $m = 0$ and $(\overline{N}, q, 1)$ is the (\overline{N}, q) method. If $q(x) = 1$ on \mathbf{R}_+ , then (\overline{N}, q, m) method is the Hölder method of order m and $(\overline{N}, q, 1)$ method is the Cesàro summability method $(C, 1)$.

It is well known that condition $Q(x) \rightarrow \infty$ as $x \rightarrow \infty$ is a necessary and sufficient condition that the existence of the integral

$$(1.3) \quad \int_0^\infty s(x)dx = L$$

implies (1.2). That is, the (\overline{N}, q, m) summability method is regular, where m is a nonnegative integer. However, the converse of this implication is not always true. Notice that some suitable condition on $s(x)$ together with (1.2) may imply (1.3).

Such a condition is called a Tauberian condition and resulting theorem is said to be a Tauberian theorem.

Móricz [8] and Fekete and Móricz [6] obtained one-sided and two-sided Tauberian conditions for the weighted mean method (\overline{N}, q) of integrals. Following these works, Totur and Okur [13] proved one-sided boundedness of $v_q^{(0)}(x)$ is a Tauberian condition for the weighted mean method of summability (\overline{N}, q) of integrals. From the fact that condition $v_q^{(0)}(x) \geq -C$ implies slow decreasing of $s(x)$, Totur and Okur [13] generalized their first result and proved that slow decrease of $s(x)$ is also a Tauberian condition for (\overline{N}, q) method. For a detailed study and some interesting results related to Tauberian theorems for the weighted mean method of summability, we refer the reader to Borwein and Kratz [1], Çanak and Totur [2], Çanak and Totur [3], Çanak and Totur [4], Özsaraç and Çanak [9], Sezer and Çanak [10], Tietz and Zeller [11] and Totur and Çanak [12], etc.

In this paper, one- and two-sided Tauberian conditions in terms of the generating function and its generalizations for summable integrals by m -th iteration of weighted means of real- or complex-valued functions have been obtained, respectively. Some classical type Tauberian theorems given for Cesàro summability $(C, 1)$ and weighted mean method of summability (\overline{N}, q) have been extended and generalized.

2. Main results

For the main results of the paper, we need the following definitions and notations.

Definition 2.1. ([7]) A positive function Q is called regularly varying of index $\alpha > 0$ if

$$(2.1) \quad \lim_{x \rightarrow \infty} \frac{Q(\rho x)}{Q(x)} = \rho^\alpha, \quad \rho > 0.$$

It easily follows from Definition 2.1 that for all $\rho > 1$ and sufficiently large x ,

$$(2.2) \quad \frac{\rho^\alpha}{2(\rho^\alpha - 1)} \leq \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \leq \frac{3\rho^\alpha}{2(\rho^\alpha - 1)}$$

and for all $0 < \rho < 1$ and sufficiently large x ,

$$(2.3) \quad \frac{\rho^\alpha}{2(1 - \rho^\alpha)} \leq \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \leq \frac{3\rho^\alpha}{2(1 - \rho^\alpha)}.$$

We note that if (2.1) holds, then the following equivalent conditions are clearly satisfied (see [5]):

$$(2.4) \quad \liminf_{x \rightarrow \infty} \frac{Q(x)}{Q(\rho x)} < 1, \quad \text{for every } \rho > 1$$

and

$$(2.5) \quad \liminf_{x \rightarrow \infty} \frac{Q(\rho x)}{Q(x)} < 1, \quad \text{for every } 0 < \rho < 1.$$

First, we consider real-valued functions and prove the following Tauberian theorems.

Theorem 2.1. *Let (2.1) be satisfied. If a real-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is (\overline{N}, q, m) summable to L and $v_q^{(m-1)}(x)$ is one-sided bounded, then $s(x)$ is $(\overline{N}, q, m-1)$ summable to L .*

Corollary 2.1. *([13]) Let (2.1) be satisfied. If a real-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $(\overline{N}, q, 1)$ summable to L and $v_q^{(0)}(x)$ is one-sided bounded, then $s(x)$ converges to L .*

Theorem 2.2. *Let (2.4) be satisfied. If a real-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is (\overline{N}, q, m) summable to L and $\tau_q^{(m-1)}(x)$ is slowly decreasing, then $s(x)$ is $(\overline{N}, q, m-1)$ summable to L .*

Corollary 2.2. *([13]) Let (2.4) be satisfied. If a real-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $(\overline{N}, q, 1)$ summable to L and slowly decreasing, then $s(x)$ converges to L .*

A real-valued function $s(x)$ defined on \mathbf{R}_+ is said to be slowly decreasing if

$$(2.6) \quad \lim_{\rho \rightarrow 1^+} \liminf_{x \rightarrow \infty} \min_{x \leq t \leq \rho x} (s(t) - s(x)) \geq 0.$$

Note that condition (2.6) can be equivalently reformulated as follows:

$$(2.7) \quad \lim_{\rho \rightarrow 1^-} \liminf_{x \rightarrow \infty} \min_{\rho x \leq t \leq x} (s(x) - s(t)) \geq 0.$$

Second, we consider complex-valued functions and prove the following Tauberian theorems.

Theorem 2.3. *Let (2.1) be satisfied. If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is (\overline{N}, q, m) summable to L and $v_q^{(m-1)}(x)$ is bounded, then $s(x)$ is $(\overline{N}, q, m-1)$ summable to L .*

Corollary 2.3. *Let (2.1) be satisfied. If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $(\overline{N}, q, 1)$ summable to L and $v_q^{(0)}(x)$ is bounded, then $s(x)$ converges to L .*

Theorem 2.4. *Let (2.4) be satisfied. If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is (\overline{N}, q, m) summable to L and $\tau_q^{(m-1)}(x)$ is slowly oscillating, then $s(x)$ is $(\overline{N}, q, m-1)$ summable to L .*

Corollary 2.4. *Let (2.4) be satisfied. If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $(\overline{N}, q, 1)$ summable to L and slowly oscillating, then $s(x)$ converges to L .*

A complex-valued function $s(x)$ defined on \mathbf{R}_+ is said to be slowly oscillating if

$$(2.8) \quad \lim_{\rho \rightarrow 1^+} \limsup_{x \rightarrow \infty} \max_{x \leq t \leq \rho x} |s(t) - s(x)| = 0.$$

Note that condition (2.8) can be equivalently reformulated as follows:

$$(2.9) \quad \lim_{\rho \rightarrow 1^-} \limsup_{x \rightarrow \infty} \max_{\rho x \leq t \leq x} |s(x) - s(t)| = 0.$$

3. An auxiliary result

The following two representations of $s(x) - \tau_q^{(1)}(x)$ will be needed in the proofs of our main results.

Lemma 3.1. ([13])

(i) For $\rho > 1$ and sufficiently large x ,

$$\begin{aligned} s(x) - \tau_q^{(1)}(x) &= \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left(\tau_q^{(1)}(\rho x) - \tau_q^{(1)}(x) \right) \\ &\quad - \frac{1}{Q(\rho x) - Q(x)} \int_x^{\rho x} (s(t) - s(x)) q(t) dt. \end{aligned}$$

(ii) For $0 < \rho < 1$ and sufficiently large x ,

$$\begin{aligned} s(x) - \tau_q^{(1)}(x) &= \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left(\tau_q^{(1)}(x) - \tau_q^{(1)}(\rho x) \right) \\ &\quad + \frac{1}{Q(x) - Q(\rho x)} \int_{\rho x}^x (s(x) - s(t)) q(t) dt. \end{aligned}$$

4. Proofs of main results

Proof of Theorem 2.1 Suppose that $s(x)$ is (\overline{N}, q, m) summable to L and $v_q^{(m-1)}(x)$ is one-sided bounded. By Lemma 3.1 (i), we have

$$\begin{aligned} \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) &= \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left(\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) \\ &\quad - \frac{1}{Q(\rho x) - Q(x)} \int_x^{\rho x} \left(\tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right) q(t) dt \\ &= \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left(\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) \\ &\quad - \frac{1}{Q(\rho x) - Q(x)} \int_x^{\rho x} \left(\int_x^t \frac{d}{dz} \tau_q^{(m-1)}(z) dz \right) q(t) dt. \end{aligned}$$

Since $v_q^{(m-1)}(x)$ is one-sided bounded, we get

$$\begin{aligned}
 \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) &\leq \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left(\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) \\
 &+ \frac{C}{Q(\rho x) - Q(x)} \int_x^{\rho x} \left(\int_x^t \frac{q(z)}{Q(z)} dz \right) q(t) dt \\
 &= \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left(\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) \\
 &+ \frac{C}{Q(\rho x) - Q(x)} \int_x^{\rho x} q(t) \log \frac{Q(t)}{Q(x)} dt \\
 (4.1) \qquad &= \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left(\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) + C \log \frac{Q(\rho x)}{Q(x)}.
 \end{aligned}$$

By (2.2) and (\bar{N}, q, m) summability of $s(x)$, we have

$$(4.2) \qquad \lim_{x \rightarrow \infty} \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left(\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) = 0.$$

Taking (4.2) into account in (4.1), we obtain

$$\limsup_{x \rightarrow \infty} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) \leq \limsup_{x \rightarrow \infty} \left(C \log \frac{Q(\rho x)}{Q(x)} \right) = C \log \rho^\alpha.$$

Letting $\rho \rightarrow 1^+$ in the last inequality, we have

$$(4.3) \qquad \limsup_{x \rightarrow \infty} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) \leq 0.$$

Similarly, from Lemma 3.1 (ii), we have

$$\begin{aligned}
 \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) &= \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left(\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right) \\
 &+ \frac{1}{Q(x) - Q(\rho x)} \int_{\rho x}^x \left(\tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t) \right) q(t) dt \\
 &= \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left(\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right) \\
 &+ \frac{1}{Q(x) - Q(\rho x)} \int_{\rho x}^x \left(\int_t^x \frac{d}{dz} \tau_q^{(m-1)}(z) dz \right) q(t) dt.
 \end{aligned}$$

Since $v_q^{(m-1)}(x)$ is one-sided bounded, we get

$$\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \geq \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left(\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right)$$

$$\begin{aligned}
& - \frac{C}{Q(x) - Q(\rho x)} \int_{\rho x}^x \left(\int_t^x \frac{q(z)}{Q(z)} dz \right) q(t) dt \\
& = \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left(\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right) \\
& - \frac{C}{Q(x) - Q(\rho x)} \int_{\rho x}^x q(t) \log \frac{Q(x)}{Q(t)} dt \\
(4.4) \quad & = \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left(\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right) - C \log \frac{Q(x)}{Q(\rho x)}.
\end{aligned}$$

By (2.3) and (\overline{N}, q, m) summability of $s(x)$, we obtain

$$(4.5) \quad \lim_{x \rightarrow \infty} \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left(\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right) = 0.$$

Taking (4.5) into account in (4.4), we obtain

$$\limsup_{x \rightarrow \infty} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) \geq - \liminf_{x \rightarrow \infty} \left(C \log \frac{Q(x)}{Q(\rho x)} \right) = -C \log \rho^\alpha.$$

Letting $\rho \rightarrow 1^-$ in the last inequality, we have

$$(4.6) \quad \limsup_{x \rightarrow \infty} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) \geq 0.$$

Combining (4.3) and (4.6), we obtain $s(x)$ is $(\overline{N}, q, m-1)$ summable to L . \square

Proof of Theorem 2.2 Suppose that $s(x)$ is (\overline{N}, q, m) summable to L and $\tau_q^{(m-1)}(x)$ is slowly decreasing. By Lemma 3.1 (i), we have

$$\begin{aligned}
\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) & = \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left(\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) \\
& - \frac{1}{Q(\rho x) - Q(x)} \int_x^{\rho x} \left(\tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right) q(t) dt \\
& \leq \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left(\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) \\
& - \frac{1}{Q(\rho x) - Q(x)} \int_x^{\rho x} q(t) \min_{x \leq t \leq \rho x} \left(\tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right) dt \\
& = \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left(\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) \\
(4.7) \quad & - \min_{x \leq t \leq \rho x} \left(\tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right).
\end{aligned}$$

Taking the lim sup of both sides of (4.7), we get

$$(4.8) \quad \begin{aligned} \limsup_{x \rightarrow \infty} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) &\leq \limsup_{x \rightarrow \infty} \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left(\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) \\ &\quad - \liminf_{x \rightarrow \infty} \min_{x \leq t \leq \rho x} \left(\tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right). \end{aligned}$$

By (2.4), we have

$$0 < \limsup_{x \rightarrow \infty} \frac{Q(\rho x)}{Q(\rho x) - Q(x)} = 1 + \left(\liminf_{x \rightarrow \infty} \frac{Q(\rho x)}{Q(x)} - 1 \right)^{-1} < \infty.$$

Since $s(x)$ is (\overline{N}, q, m) summable to L , the first term on the right-hand side vanishes in (4.8). From this, we obtain

$$\limsup_{x \rightarrow \infty} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) \leq - \liminf_{x \rightarrow \infty} \min_{x \leq t \leq \rho x} \left(\tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right).$$

Taking the limit of (4.8) as $\rho \rightarrow 1^+$, we have

$$(4.9) \quad \limsup_{x \rightarrow \infty} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) \leq 0.$$

Similarly, by Lemma 3.1 (ii), we have

$$(4.10) \quad \begin{aligned} \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) &= \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left(\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right) \\ &\quad + \frac{1}{Q(x) - Q(\rho x)} \int_{\rho x}^x \left(\tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t) \right) q(t) dt \\ &\geq \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left(\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right) \\ &\quad + \frac{1}{Q(x) - Q(\rho x)} \int_{\rho x}^x q(t) \min_{\rho x \leq t \leq x} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t) \right) dt \\ &= \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left(\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right) \\ &\quad + \min_{\rho x \leq t \leq x} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t) \right). \end{aligned}$$

From (4.10), we get

$$(4.11) \quad \begin{aligned} \liminf_{x \rightarrow \infty} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) &\geq \liminf_{x \rightarrow \infty} \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left(\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right) \\ &\quad + \liminf_{x \rightarrow \infty} \min_{\rho x \leq t \leq x} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t) \right). \end{aligned}$$

By (2.4), we have

$$0 < \liminf_{x \rightarrow \infty} \frac{Q(\rho x)}{Q(x) - Q(\rho x)} = \left(\limsup_{x \rightarrow \infty} \frac{Q(x)}{Q(\rho x)} - 1 \right)^{-1} < \infty.$$

Since $s(x)$ is (\overline{N}, q, m) summable to L , the first term on the right-hand side vanishes in (4.11). From this, we obtain

$$\liminf_{x \rightarrow \infty} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) \geq \liminf_{x \rightarrow \infty} \min_{\rho x \leq t \leq x} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t) \right).$$

Taking the limit of (4.11) as $\rho \rightarrow 1^-$, we have

$$(4.12) \quad \liminf_{x \rightarrow \infty} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) \geq 0.$$

Combining (4.9) and (4.12), we obtain $s(x)$ is $(\overline{N}, q, m-1)$ summable to L . \square

Proof of Theorem 2.3 Suppose that $s(x)$ is (\overline{N}, q, m) summable to L and $v_q^{(m-1)}(x)$ is bounded. By Lemma 3.1 (i), we have

$$\begin{aligned} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| &\leq \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left| \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right| \\ &+ \frac{1}{Q(\rho x) - Q(x)} \int_x^{\rho x} \left| \tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right| q(t) dt \\ &= \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left| \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right| \\ &+ \frac{1}{Q(\rho x) - Q(x)} \int_x^{\rho x} \left| \int_x^t \frac{d}{dz} \tau_q^{(m-1)}(z) dz \right| q(t) dt. \end{aligned}$$

Since $v_q^{(m-1)}(x)$ is bounded, we get

$$\begin{aligned} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| &\leq \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left| \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right| \\ &+ \frac{C}{Q(\rho x) - Q(x)} \int_x^{\rho x} \left| \int_x^t \frac{q(z)}{Q(z)} dz \right| q(t) dt \\ &= \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left| \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right| \\ &+ \frac{C}{Q(\rho x) - Q(x)} \int_x^{\rho x} q(t) \log \frac{Q(t)}{Q(x)} dt \\ (4.13) \quad &\leq \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left| \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right| + C \log \frac{Q(\rho x)}{Q(x)}. \end{aligned}$$

By (2.2) and (\overline{N}, q, m) summability of $s(x)$, we have

$$\lim_{x \rightarrow \infty} \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left| \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right| = 0.$$

Taking the lim sup of both sides of (4.13) gives

$$\limsup_{x \rightarrow \infty} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| \leq \limsup_{x \rightarrow \infty} \left(C \log \frac{P(\rho x)}{Q(x)} \right) = C \log \rho^\alpha.$$

Letting $\rho \rightarrow 1^+$ in last inequality, we have

$$(4.14) \quad \limsup_{x \rightarrow \infty} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| \leq 0.$$

Similarly, from Lemma 3.1 (ii), we have

$$\begin{aligned} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| &\leq \frac{P(\rho x)}{Q(x) - Q(\rho x)} \left| \tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right| \\ &+ \frac{1}{Q(x) - Q(\rho x)} \int_{\rho x}^x \left| \tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t) \right| q(t) dt \\ &= \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left| \tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right| \\ &+ \frac{1}{Q(x) - Q(\rho x)} \int_{\rho x}^x \left| \int_t^x \frac{d}{dz} \tau_q^{(m-1)}(z) dz \right| q(t) dt. \end{aligned}$$

Since $v_q^{(m-1)}(x)$ is bounded, we get

$$\begin{aligned} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| &\leq \frac{P(\rho x)}{Q(x) - Q(\rho x)} \left| \tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right| \\ &+ \frac{C}{Q(x) - Q(\rho x)} \int_{\rho x}^x \left| \int_t^x \frac{p(z)}{P(z)} dz \right| q(t) dt \\ &= \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left| \tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right| \\ &+ \frac{C}{Q(x) - Q(\rho x)} \int_{\rho x}^x q(t) \log \frac{Q(x)}{Q(t)} dt \\ (4.15) \quad &\leq \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left| \tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right| + C \log \frac{Q(x)}{Q(\rho x)}. \end{aligned}$$

By (2.3) and (\overline{N}, p, m) summability of $s(x)$, we have

$$\lim_{x \rightarrow \infty} \frac{Q(\rho x)}{P(x) - Q(\rho x)} \left| \tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right| = 0.$$

From (4.15), we get

$$\limsup_{x \rightarrow \infty} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| \leq \limsup_{x \rightarrow \infty} \left(C \log \frac{Q(x)}{Q(\rho x)} \right) = C \log \rho^\alpha.$$

Letting $\rho \rightarrow 1^-$ in last inequality, we have

$$(4.16) \quad \limsup_{x \rightarrow \infty} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| \leq 0.$$

From either (4.14) or (4.16), we conclude $s(x)$ is $(\overline{N}, q, m-1)$ summable to L . \square

Proof of Theorem 2.4 Suppose that $s(x)$ is (\overline{N}, q, m) summable to L and $\tau_q^{(m-1)}(x)$ is slowly oscillating. By Lemma 3.1 (i), we have

$$\begin{aligned} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| &= \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left| \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right| \\ &\quad + \frac{1}{Q(\rho x) - Q(x)} \int_x^{\rho x} \left| \tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right| q(t) dt \\ &\leq \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left| \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right| \\ &\quad + \frac{1}{Q(\rho x) - Q(x)} \int_x^{\rho x} q(t) \max_{x \leq t \leq \rho x} \left(\left| \tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right| \right) dt \\ &\leq \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left| \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right| \\ (4.17) \quad &\quad + \max_{x \leq t \leq \rho x} \left| \tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right|. \end{aligned}$$

From (4.17), we get

$$\begin{aligned} \limsup_{x \rightarrow \infty} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| &\leq \limsup_{x \rightarrow \infty} \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \left| \tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right| \\ (4.18) \quad &\quad + \limsup_{x \rightarrow \infty} \max_{x \leq t \leq \rho x} \left| \tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right|. \end{aligned}$$

By (2.4), we have

$$0 < \limsup_{x \rightarrow \infty} \frac{Q(\rho x)}{Q(\rho x) - Q(x)} = 1 + \left(\liminf_{x \rightarrow \infty} \frac{Q(\rho x)}{Q(x)} - 1 \right)^{-1} < \infty.$$

Since $s(x)$ is (\overline{N}, q, m) summable to L , the first term on the right side vanishes in (4.18). From this, we obtain

$$\limsup_{x \rightarrow \infty} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| \leq \limsup_{x \rightarrow \infty} \max_{x \leq t \leq \rho x} \left| \tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right|.$$

Taking the limit of (4.18) as $\rho \rightarrow 1^+$, we have

$$(4.19) \quad \limsup_{x \rightarrow \infty} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| \leq 0.$$

Similarly, by Lemma 3.1 (ii), we have

$$\begin{aligned} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| &= \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left| \tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right| \\ &+ \frac{1}{Q(x) - Q(\rho x)} \int_{\rho x}^x \left| \tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t) \right| q(t) dt \\ &\leq \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left| \tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right| \\ &+ \frac{1}{Q(x) - Q(\rho x)} \int_{\rho x}^x q(t) \max_{\rho x \leq t \leq x} \left(\left| \tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t) \right| \right) dt \\ &\leq \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left| \tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right| \\ (4.20) \quad &+ \max_{\rho x \leq t \leq x} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t) \right|. \end{aligned}$$

From (4.20), we get

$$\begin{aligned} \limsup_{x \rightarrow \infty} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| &\leq \limsup_{x \rightarrow \infty} \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left| \tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right| \\ (4.21) \quad &+ \limsup_{x \rightarrow \infty} \max_{\rho x \leq t \leq x} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t) \right|. \end{aligned}$$

By (2.4), we have

$$0 < \liminf_{x \rightarrow \infty} \frac{Q(\rho x)}{Q(x) - Q(\rho x)} = \left(\limsup_{x \rightarrow \infty} \frac{Q(x)}{Q(\rho x)} - 1 \right)^{-1} < \infty.$$

Since $s(x)$ is (\overline{N}, q, m) summable to L , the first term on the right-hand side vanishes in (4.21). From this, we obtain

$$\limsup_{x \rightarrow \infty} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| \leq \limsup_{x \rightarrow \infty} \max_{\rho x \leq t \leq x} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t) \right|.$$

Taking the limit of (4.21) as $\rho \rightarrow 1^-$, we have

$$(4.22) \quad \limsup_{x \rightarrow \infty} \left| \tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right| \leq 0.$$

From either (4.19) or (4.22), we conclude $s(x)$ is $(\overline{N}, q, m-1)$ summable to L . \square

5. Conclusion

In this paper, we introduce Tauberian conditions in terms of the generator and its generalizations for summable integrals by m -th iteration of weighted means of real- or complex-valued functions, respectively. Tauberian conditions for summable double integrals by m -th iteration of weighted means of real- or complex-valued functions will be illustrated in a forthcoming work.

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